

Perturbations in Massive Gravity Cosmology

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ABSTRACT: We study cosmological perturbations for a ghost free massive gravity theory formulated with a dynamical extra metric that is needed to massive deform GR. In this formulation FRW background solutions fall in two branches. In the dynamics of perturbations around the first branch solutions, no extra degree of freedom with respect to GR is present at linearized level, likewise what is found in the Stuckelberg formulation of massive gravity where the extra metric is flat and non dynamical. In the first branch, perturbations are probably strongly coupled. On the contrary, for perturbations around the second branch solutions all expected degrees of freedom propagate. While tensor and vector perturbations of the physical metric that couples with matter follow closely the ones of GR, scalars develop an exponential Jeans-like instability on sub-horizon scales. On the other hand, around a de Sitter background there is no instability. We argue that one could get rid of the instabilities by introducing a mirror dark matter sector minimally coupled to only the second metric.

KEYWORDS: Massive Gravity, Cosmological Perturbations

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1 Introduction

Recently, there has been a renewed interest in the search of a modified theory of gravity at large distances through a massive deformation of GR (see for a recent review [1]). A great deal of effort was devoted to extend at the nonlinear level [2] the seminal work of Fierz and Pauli (FP) [3]. The FP theory is defined at linearized level and is plagued by a number of diseases. In particular, the modification of the Newtonian potential is not continuous when the mass m^2 vanishes, giving a large correction (25%) to the light deflection from the sun that is experimentally excluded [4]. A possible way to circumvent the the discontinuity problem is to suppose that [5] the linearized approximation breaks down near a massive object like the sun and an improved perturbative expansion. In addition,

FP is problematic as an effective theory. Regarding FP as a gauge theory where the gauge symmetry is broken by a explicit mass term m , one would expect a cutoff $\Lambda_2 \sim mg^{-1} = (mM_{pl})^{1/2}$, however the real cutoff is $\Lambda_5 = (m^4 M_{pl})^{1/5}$ or $\Lambda_3 = (m^2 M_{pl})^{1/3}$, much lower than Λ_2 [6]. A would-be Goldstone mode is responsible for the extreme UV sensitivity of the FP theory, that becomes totally unreliable in the absence of proper UV completion. Recently it was shown that there exists a non linear completion of the FP theory [7] that is free of ghosts up to the fourth order [7], avoiding the presence of the Boulware-Deser instability [8]. Then the propagation of only five degrees of freedom (DoF) was generalized to all orders in [9]; this was shown also in the Stuckelberg language in [10].

Quite naturally massive gravity leads to bigravity. Indeed, any massive deformation, obtained by adding to the Einstein-Hilbert action a non-derivative self-coupling for the metric g , requires the introduction of an additional metric \tilde{g} that may be a fixed external field, or be a dynamical one. When \tilde{g} is non-dynamical we are in the framework of æther-like theories; on the other hand if it is dynamical we enter in the realm of bigravity [11] that was originally introduced by Isham, Salam and Strathdee [12]. The need for a second dynamical metric also follows from rather general grounds. Indeed, it was shown in [13] that in the case of non singular static spherically symmetric geometry with the additional property that the two metrics are diagonal in the same coordinate patch, a Killing horizon for g must also be a Killing horizon for \tilde{g} ; see [14] for a concrete example. Actually it turns out that the off diagonal solutions show no modification of gravity at large distance [15, 16]. Also cosmology calls for the bigravity formulation of massive gravity. When the second metric is static there is no homogeneous spatially flat FRW solution [17–19], on the contrary in the bigravity formulation flat FRW homogeneous solutions do exist [20–22]. See also [23] for a different approach to cosmology in massive gravity. In this paper we study perturbations around the FRW background solutions found in [20].

The outline of the paper is the following. After a brief introduction to the bigravity formulation of massive gravity in section 2, the FRW solutions are reviewed in section 3. The perturbed FRW geometry is introduced in section 4 and the perturbed Einstein equations are given in section 5. The dynamics of the perturbations in various cases are studied in sections 6, 7 and 8. Section 9 contains our conclusions.

2 Massive Gravity and Bigravity

Any modification of GR that turns a massless graviton into a massive one calls for additional DoF. An elegant way to provide them is to work with an extra tensor $\tilde{g}_{\mu\nu}$. When coupled to the standard metric $g_{\mu\nu}$, it allows to build non-trivial diff-invariant operators that lead to mass terms when expanded around a background. Consider the action

$$S = \int d^4x \left\{ \sqrt{\tilde{g}} \kappa M_{pl}^2 \tilde{\mathcal{R}} + \sqrt{g} \left[M_{pl}^2 (\mathcal{R} - 2m^2 V) + L_{\text{matt}} \right] \right\}, \quad (2.1)$$

where $R(g_i)$ are the corresponding Ricci scalars and the interaction potential V is a scalar function of the tensor $X^\mu_\nu = g^{\mu\alpha} \tilde{g}_{\alpha\nu}$. Matter is minimally coupled to g and it is described by L_{matt} . The constant κ controls the relative size of the strength of gravitational interactions in the two sectors, while m sets the scale of the graviton mass. The action (2.1) brings us into the realm of bigravity theories, whose study started in the '60 (see [11] for early references). An action of the form (2.1) can be also viewed as the effective theories for the low lying Kaluza-Klein modes in brane world models [11]. The massive deformation is encoded in the non derivative coupling between $g_{\mu\nu}$ and the extra tensor field $\tilde{g}_{\mu\nu}$. Clearly the action is invariant under diffeomorphisms, which transform the two fields in the same

way (diagonal diffs). Taking the limit $\kappa \rightarrow \infty$, the second metric decouples, and gets effectively frozen to a fixed background value so that the “relative” diffeomorphisms are effectively broken, as far as the first metric is concerned. Depending on the background value of $\tilde{g}_{\mu\nu}$ one can explore both the Lorentz-invariant (LI) and the Lorentz-breaking (LB) phases of massive gravity [25] [26]. When the second metric is dynamical this is determined by its asymptotic properties, as discussed below. In this case notice that $\tilde{g}_{\mu\nu}$ is determined by its equations of motion (for any finite \tilde{M}_{pl}) so that we will be working always with consistent and dynamically determined backgrounds. The role played by $\tilde{g}_{\mu\nu}$ is very similar to the Higgs field, its dynamical part restores gauge invariance and its background value determines the realization of the residual symmetries.

The modified Einstein equations can be written as¹

$$E_\nu^\mu + Q_{1\nu}^\mu = \frac{1}{2 M_{pl}^2} T_\nu^\mu \quad (2.2)$$

$$\kappa \tilde{E}_\nu^\mu + Q_{2\nu}^\mu = 0 ; \quad (2.3)$$

where we have defined Q_1 and Q_2 as effective energy-momentum tensors induced by the interaction term. The only invariant tensor that can be written without derivatives out of g and \tilde{g} is $X_\nu^\mu = g_1^{\mu\alpha} \tilde{g}_{\alpha\nu}$ [11]. The ghost free potential [7]² V is a special scalar function of $Y_\nu^\mu = (\sqrt{X})_\nu^\mu$ given by

$$V = \sum_{n=0}^4 a_n V_n , \quad n = 0 \dots 4, \quad (2.4)$$

where The V_n are the symmetric polynomials of Y

$$\begin{aligned} V_0 &= 1 & V_1 &= \tau_1 , & V_2 &= \tau_1^2 - \tau_2 , & V_3 &= \tau_1^3 - 3 \tau_1 \tau_2 + 2 \tau_3 , \\ V_4 &= \tau_1^4 - 6 \tau_1^2 \tau_2 + 8 \tau_1 \tau_3 + 3 \tau_2^2 - 6 \tau_4 , \end{aligned} \quad (2.5)$$

with $\tau_n = \text{tr}(Y^n)$. In [27] it was shown that in the bimetric formulation the potential V is ghost free. We have that

$$Q_{1\nu}^\mu = m^2 [V \delta_\nu^\mu - (V' Y)_\nu^\mu] \quad (2.6)$$

$$Q_{2\nu}^\mu = m^2 q^{-1/2} (V' Y)_\nu^\mu , \quad (2.7)$$

where $(V')_\nu^\mu = \partial V / \partial Y_\nu^\mu$ and $q = \det X = \det(\tilde{g}) / \det(g)$.

The canonical analysis [9] shows that in general 7 DoF propagate; around a Minkowski background, 5 can be associated to a massive spin two graviton and the remaining 2 to a massless spin two graviton.

3 FRW Solutions in Massive Gravity

Let us review the FRW background solutions in massive gravity [20] that are of the form

$$\begin{aligned} ds^2 &= a^2(\tau) (-d\tau^2 + dr^2 + r^2 d\Omega^2) = \bar{g}_{1\mu\nu} dx^\mu dx^\nu \\ \tilde{ds}^2 &= \omega^2(\tau) [-c^2(\tau) d\tau^2 + dr^2 + r^2 d\Omega^2] = \bar{g}_{2\mu\nu} dx^\mu dx^\nu . \end{aligned} \quad (3.1)$$

It is convenient to define the standard Hubble parameters for the two metrics

$$\mathcal{H} = \frac{da}{d\tau} \frac{1}{a} \equiv \frac{a'}{a} = H a , \quad \mathcal{H}_\omega = \frac{\omega'}{\omega} = H_\omega \omega , \quad \xi = \frac{\omega}{a} . \quad (3.2)$$

Solutions fall in two branches depending on how the Bianchi identities are realized.

¹When not specified, indices of tensors related with $g(\tilde{g})$ are raised/lowered with $g(\tilde{g})$

²A very similar potential having the same form but with X instead of $X^{1/2}$ was considered in [15].

- In branch one, $\xi = \bar{\xi}$ is constant and satisfies the following algebraic equation

$$f_2(\bar{\xi}) = 0 \quad \text{with } f_2(\xi) = 6 a_3 \xi^2 + 4 a_2 \xi + a_1. \quad (3.3)$$

As a consequence, the Hubble parameter \mathcal{H} of g and the one of \tilde{g} , \mathcal{H}_ω coincide and

$$\frac{\mathcal{H}^2}{a^2} = \frac{8\pi G}{3} (\rho + \Lambda_1), \quad \Lambda_1 = \frac{m^2}{8\pi G} [a_0 - 6 \bar{\xi}^2 (2 a_3 \bar{\xi} + a_2)]. \quad (3.4)$$

In this branch the effect of the mass deformation is to induce an effective cosmological constant and

$$c = \sqrt{\frac{\Lambda_1 + \rho}{\Lambda_2}} \quad \text{with } \Lambda_2 = \frac{m^2}{4\pi G \kappa} [6 \bar{\xi} (2 a_4 \bar{\xi} + a_3) + a_2]. \quad (3.5)$$

This is not very surprising, indeed, the constraint (3.3), in the spherically symmetric case, leads to a branch of solutions with no modification of gravity, being the graviton mass zero around a flat background [16].

- In branch two, ξ is not constant and the Bianchi identities are realized in the form

$$c = \frac{H_\omega}{H} \xi = \frac{\mathcal{H}_\omega}{\mathcal{H}}, \quad \xi' = (c - 1) \mathcal{H} \xi \quad \text{with } c > 0, \quad (3.6)$$

and

$$\frac{3\mathcal{H}^2}{a^2} = 8\pi G \rho + m^2 (6a_3 \xi^3 + 6a_2 \xi^2 + 3a_1 \xi + a_0). \quad (3.7)$$

The ratio ξ of the two scale parameters satisfies the equation

$$m^2 \left[\xi^2 \left(\frac{8 a_4}{\kappa} - 2 a_2 \right) + \xi \left(\frac{6 a_3}{\kappa} - a_1 \right) + \frac{a_1}{3 \kappa \xi} + \frac{2 a_2}{\kappa} - 2 a_3 \xi^3 - \frac{a_0}{3} \right] = \frac{8\pi G \rho}{3}. \quad (3.8)$$

In the expanding universe, the ratio $\frac{8\pi G \rho}{m^2}$ scales rapidly with redshift z , $1+z \equiv a(\tau_{today})/a(\tau)$. Indeed, assuming that the mass scale m is related to the present cosmological constant as $m^2 M_{pl}^2 \propto \Lambda$ and that for matter with an equation of state $p = w \rho$, $\rho = \rho_0 (1+z)^{3(w+1)}$, we have

$$\frac{8\pi G \rho}{m^2} \sim \frac{\Omega_m}{\Omega_\Lambda} z^{3(w+1)} \gg 1 \quad \text{at early times.} \quad (3.9)$$

This quantity gets very large already at redshift $z \sim 10$, much later than the radiation era ($z \sim 10^4$). Thus, eq.(3.8) can be satisfied only in two regimes: for large or small values of ξ ³. The large ξ regime is physically uninteresting because $c < 0$ when matter has $w > 0$ [20]. In the small ξ regime, cosmology is very similar to the standard one; for $a_1 > 0$ we have

$$\xi = \frac{a_1 m^2}{8\pi G \kappa \rho} + \mathcal{O} \left(\frac{m^2}{G \rho} \right)^2 \sim z^{-3(w+1)}; \quad (3.10)$$

$$\rho + \rho_g = \rho \left(1 + \mathcal{O} \left(\frac{m^2}{G \rho} \right) \right); \quad (3.11)$$

$$w_{eff} = w + \mathcal{O} \left(\frac{m^2}{G \rho} \right), \quad c = (4 + 3w) + \mathcal{O} \left(\frac{m^2}{G \rho} \right). \quad (3.12)$$

³In presence of “mirror” matter with energy density $\tilde{\rho}$ minimally coupled with the second metric \tilde{g} , eq.(3.8) gets modified as $\rho \rightarrow \rho - \xi^2 \tilde{\rho}/\kappa$ and the small-large ξ regime solutions can be evaded as soon as $\xi^2 \sim \kappa \rho/\tilde{\rho}$, when $m \rightarrow 0$.

Once the matter is so diluted that ρ is negligible in (3.8) the system falls in the fixed point region and ξ is almost constant and the universe enters in a late time dS phase. The analysis is identical when the *same* spatial curvature k_c is introduced in (3.1)⁴ for both metrics.

Let us now discuss the differences with the frozen metric approach where \tilde{g} is non dynamical. Formally, the non-dynamical limit corresponds to $\kappa \rightarrow \infty$. To make contact with the existing literature, we take \tilde{g} equivalent to the Minkowski flat metric implying clearly that c and ω cannot be arbitrary. Indeed, imposing that the Riemann curvature tensor of \tilde{g} vanishes we get that $k_c < 0$ and

$$c = \frac{\mathcal{H}_\omega}{\sqrt{-k_c}}. \quad (3.13)$$

Thus, flat FRW cosmology with frozen second metric exists only with a negative non-vanishing spatial curvature [17]-[20]. When (3.13) holds, Bianchi identities can be realized only within branch one, leading to eq.(3.3). That is why FRW perturbations in the Stuckelberg formalism are stuck into branch one that is rather problematic.

4 Perturbed FRW Geometry

Let us now consider the perturbations of the FRW background (3)

$$g_{\mu\nu} = \bar{g}_{1\mu\nu} + a^2 h_{1\mu\nu}, \quad \tilde{g}_{\mu\nu} = \bar{g}_{2\mu\nu} + \omega^2 h_{2\mu\nu}. \quad (4.1)$$

parametrized as follows

$$\begin{aligned} h_{100} &\equiv -2A_1, & h_{200} &\equiv -2c^2 A_2 \\ h_{1/2\,0i} &\equiv \mathcal{C}_{1/2\,i} - \partial_i B_{1/2}, & \partial^i \mathcal{V}_{1/2\,i} &= \partial^i \mathcal{C}_{1/2\,i} = \partial^j h^{TT}_{1/2\,ij} = \delta^{ij} h^{TT}_{1/2\,ij} = 0, \\ h_{1/2\,ij} &\equiv h^{TT}_{1/2\,ij} + \partial_i \mathcal{V}_{1/2\,j} + \partial_j \mathcal{V}_{1/2\,i} + 2\partial_i \partial_j E_{1/2} + 2\delta_{ij} F_{1/2}. \end{aligned} \quad (4.2)$$

Spatial indices are raised/lowered using the spatial flat metric.

Under a gauge transformation generated by ζ^μ the metric perturbation transforms

$$\begin{aligned} \delta h_{1\mu\nu} &= a^{-2} (\zeta^\alpha \partial_\alpha \bar{g}_{1\mu\nu} + \bar{g}_{1\alpha\nu} \partial_\mu \zeta^\alpha + \bar{g}_{1\mu\alpha} \partial_\nu \zeta^\alpha), \\ \delta h_{2\mu\nu} &= \omega^{-2} (\zeta^\alpha \partial_\alpha \bar{g}_{2\mu\nu} + \bar{g}_{2\alpha\nu} \partial_\mu \zeta^\alpha + \bar{g}_{2\mu\alpha} \partial_\nu \zeta^\alpha). \end{aligned} \quad (4.3)$$

and for the corresponding components

$$\begin{aligned} \delta A_1 &= \mathcal{H} \zeta^0 + \zeta^{0'}, & \delta B_1 &= \zeta^0 - \zeta', & \delta E_1 &= \zeta, & \delta F_1 &= \mathcal{H} \zeta^0; \\ \delta A_2 &= \mathcal{H}_\beta \zeta^0 + \zeta^{0'}, & \delta B_2 &= c^2 \zeta^0 - \zeta', & \delta E_2 &= \zeta, & \delta F_2 &= \mathcal{H}_\omega \zeta^0; \\ \delta \mathcal{C}_{1/2\,i} &= \zeta_T^{i'}, & \delta \mathcal{V}_{1/2\,i} &= \zeta_T^i, & \delta h^{TT}_{1/2\,ij} &= 0; \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} \zeta^i &= \zeta_T^i + \partial_i \zeta, & \zeta &= \Delta^{-1} \partial_i \zeta^i, \\ \mathcal{H}_\beta &= \frac{(c\omega)'}{(c\omega)} = \frac{c'}{c} + \omega \mathcal{H}_\omega. \end{aligned} \quad (4.5)$$

⁴The spatial curvatures must be equal for consistency [20].

In the scalar sector we have 8 fields and two independent gauge transformations, as a result we can form 6 independent gauge invariant scalar combinations that we chose to be

$$\begin{aligned}\Psi_1 &= A_1 - \mathcal{H} \Xi_1 - \Xi'_1 & \Psi_2 &= A_2 + c^{-2} \left(\frac{c'}{c} - \mathcal{H}_\omega \right) \Xi_2 - \frac{\Xi'_2}{c^2} \\ \Phi_1 &= F_1 - \mathcal{H} \Xi_1, & \Phi_2 &= F_2 - \mathcal{H}_\omega \frac{\Xi_2}{c^2}, \\ \mathcal{E} &= E_1 - E_2, & \mathcal{B}_1 &= B_2 - c^2 B_1 + (1 - c^2) E'_1,\end{aligned}\tag{4.6}$$

where $\Xi_{1/2} = B_{1/2} + E'_{1/2}$. The following additional gauge invariant fields will be useful to write in a compact form the perturbed Einstein equations

$$\begin{aligned}\mathcal{F}_1 &= F_2 - F_1 + (\mathcal{H} - \mathcal{H}_\omega) \Xi_1, & \mathcal{F}_2 &= F_2 - F_1 + (\mathcal{H} - \mathcal{H}_\omega) \frac{\Xi_2}{c^2}, \\ \mathcal{B}_2 &= B_2 - c^2 B_1 + (1 - c^2) E'_2, \\ \mathcal{A}_1 &= c(A_2 - A_1) + [c(\mathcal{H} - \mathcal{H}_\omega) - c'] \Xi_1, \\ \mathcal{A}_2 &= c(A_2 - A_1) + [c(\mathcal{H} - \mathcal{H}_\omega) - c'] \frac{\Xi_2}{c^2}.\end{aligned}\tag{4.7}$$

The fields $\mathcal{F}_{1/2}$, $\mathcal{A}_{1/2}$ and \mathcal{B}_2 can be expressed in terms of the ones in (4.6), as it is shown in Appendix A. In the matter sector, we define the following gauge invariant perturbed pressure and density

$$\delta\rho_{gi} = \delta\rho - \Xi_1 \rho', \quad \delta p_{gi} = \delta p - \Xi_1 p'.\tag{4.8}$$

For matter, together with pressure and density perturbation, there is also the perturbed 4-velocity u^μ that consists of a scalar part v and a vector part δz_i

$$\begin{aligned}u^\mu &= \bar{u}^\mu + \delta u^\mu, & u^\mu u^\nu g_{\mu\nu} &= -1, & \delta u^0 &= -a^{-1} A_1; \\ \delta u_i &= a (\partial_i v - \partial_i B_1 + \delta z_i + \mathcal{C}_{1i}).\end{aligned}\tag{4.9}$$

The corresponding gauge invariant quantity are defined as

$$u_s = v + E'_1, \quad \delta v_i = \delta z_i + \mathcal{C}_{1i}.\tag{4.10}$$

The conservation of the matter EMT leads to a set of differential relations; for scalar matter perturbations we have

$$\delta\rho'_{gi} = (1 + w) [\rho (k^2 u_s - 3 \Phi'_1) - 3 \mathcal{H} \delta\rho];\tag{4.11}$$

$$u'_s = (3w - 1) u_s \mathcal{H} - \frac{w}{(1 + w)} \frac{\delta\rho_{gi}}{\rho} - \Psi_1;\tag{4.12}$$

while for vector matter perturbations

$$\delta v'_i = \delta v_i (3w - 1) \mathcal{H}.\tag{4.13}$$

In the vector sector we have 4 fields and 1 gauge transformation; thus, we can form 3 independent gauge invariant vector perturbations

$$V_{1/2i} = \mathcal{C}_{1/2i} - \mathcal{V}'_{1/2i}, \quad \chi_i = \mathcal{C}_{1i} - \mathcal{C}_{2i}.\tag{4.14}$$

5 Perturbed Einstein Equations

Let us start with the scalar sector. The leading order perturbed Einstein equations for g are

$$2\Delta\Phi_1 + 6\mathcal{H}(\Psi_1\mathcal{H} - \Phi'_1) + a^2 m^2 f_2(3\mathcal{F}_1 - \Delta\mathcal{E}) = -8\pi a^2 G \delta\rho_{\text{gi}}; \quad (5.1)$$

$$\partial_i \left[2\Psi_1\mathcal{H} - 2\Phi'_1 + \frac{a^2 m^2 \mathcal{B}_1 f_2}{(c+1)} + 8\pi G a^2 (p + \rho) u_s \right] = 0; \quad (5.2)$$

$$\begin{aligned} (\partial_i \partial_j - \delta_{ij} \Delta) (a^2 f_1 m^2 \mathcal{E} - \Phi_1 - \Psi_1) + \delta_{ij} [m^2 a^2 (2 f_1 \mathcal{F}_1 + f_2 \mathcal{A}_1) + 2\Psi_1 (\mathcal{H}^2 + 2\mathcal{H}')] \\ - 2\Phi''_1 - 2\mathcal{H}(2\Phi'_1 - \Psi'_1)] = 8\pi G a^2 \delta_{ij} \delta p_{\text{gi}}, \end{aligned} \quad (5.3)$$

where

$$f_1 = \xi [2\xi (3a_3 c \xi + a_2 (c+1)) + a_1], \quad f_2 = \xi (6a_3 \xi^2 + 4a_2 \xi + a_1). \quad (5.4)$$

For the metric \tilde{g} we have

$$2c^2 \Delta\Phi_2 + 6\mathcal{H}_\omega (\Psi_2\mathcal{H}_\omega - \Phi'_2) + \frac{m^2 a^2 f_2}{\kappa \xi^2} c^2 (\Delta\mathcal{E} - 3\mathcal{F}_2) = 0; \quad (5.5)$$

$$\partial_i \left[2c (\Psi_2\mathcal{H}_\omega - \Phi'_2) - \frac{m^2 a^2 f_2}{\kappa \xi^2 (1+c)} \mathcal{B}_2 \right] = 0; \quad (5.6)$$

$$\begin{aligned} -c (\partial_i \partial_j - \delta_{ij} \Delta) \left[\frac{a^2 f_1 m^2}{\kappa \xi^2} \mathcal{E} + c (\Phi_2 + \Psi_2) \right] + \delta_{ij} \left[\frac{m^2 a^2}{\kappa \xi^2} (2c f_1 \mathcal{F}_2 + f_2 \mathcal{A}_2) + \right. \\ \left. 2 \left(\mathcal{H}_\omega^2 + 2\mathcal{H}'_\omega - 2\frac{c'}{c} \mathcal{H}_\omega \right) \Psi_2 - 2\Phi''_2 + 2 \left(\frac{c'}{c} - 2\mathcal{H}_\omega \right) \Phi'_2 + 2\mathcal{H}_\omega \Psi'_2 \right] = 0. \end{aligned} \quad (5.7)$$

For the vector sector the perturbed Einstein equations are

$$\frac{\Delta V_{1i}}{2a^2} - 8\pi G (\rho + p) \delta v_i - \frac{m^2}{(1+c)} f_2 \chi_i = 0; \quad (5.8)$$

$$\partial_{(i} V'_{1j)} + 2\mathcal{H} \partial_{(i} V_{1j)} = m^2 a^2 f_1 \partial_{(i} \mathcal{V}_{12j)}; \quad (5.9)$$

$$\frac{\Delta V_{2i}}{2a^2 c} + \frac{m^2 f_2}{(1+c) \kappa \xi^2} \chi_i = 0; \quad (5.10)$$

$$\partial_{(i} V'_{2j)} + \left[2 \left(\mathcal{H} + \frac{\xi'}{\xi} \right) - \frac{c'}{c} \right] \partial_{(i} V_{2j)} + \frac{m^2 a^2 c f_1}{\kappa \xi^2} \partial_{(i} \mathcal{V}_{12j)} = 0; \quad (5.11)$$

where

$$\mathcal{V}_{12i} = \mathcal{V}_{1i} - \mathcal{V}_{2i}, \quad V_{12i} = V_{1i} - V_{2i}. \quad (5.12)$$

Notice that $V_{12i} = \chi_i - \mathcal{V}'_{12i}$.

Finally, for the tensor perturbations we obtain

$$h^{TT''}_{1ij} + 2\mathcal{H} h^{TT'}_{1ij} - \Delta h^{TT}_{1ij} + m^2 a^2 f_1 (h^{TT}_{1ij} - h^{TT}_{2ij}) = 0; \quad (5.13)$$

$$h^{TT''}_{2ij} + \left[2 \left(\mathcal{H} + \frac{\xi'}{\xi} \right) - \frac{c'}{c} \right] h^{TT'}_{2ij} - c^2 \Delta h^{TT}_{2ij} - \frac{m^2 f_1 c}{\kappa \xi^2} a^2 (h^{TT}_{1ij} - h^{TT}_{2ij}) = 0. \quad (5.14)$$

Now that we have at our disposal the full set of equations, we can study perturbations around background solutions in the branch one and two. In the following we will often use the Fourier transform of perturbations with the respect to x^i , the corresponding 3-momentum will be k^i and $k^2 = k^i k_i$. To keep notation as simple as possible the symbol of the Fourier transform will be understood.

6 Branch One Perturbations

In this case $\xi = \bar{\xi}$ is a non-vanishing constant such that $f_2(\bar{\xi}) = 0$, then $\mathcal{H}_\omega = \mathcal{H}$. From (5.6) we can express Ψ_2 in terms of Φ_2 , then from (5.5) we get that $\Psi_2 = \Phi_2 = 0$. Now, using (5.7) we have that $\mathcal{E} = \mathcal{F}_2 = 0$ and from the relations in Appendix A, also $\mathcal{F}_1 = 0$. At this point it is straightforward to show that using the equations for the scalar perturbations of g we get

$$[-w \Delta + (3w + 1)\mathcal{H}^2 + 2\mathcal{H}'] \Psi_1 + 3(w + 1)\mathcal{H}\Psi_1' + \Psi_1'' = 0 \quad (6.1)$$

Not surprisingly, this equation describes the very same perturbations of GR in the presence of a fluid with an equation of state w .

Also for vectors, being $f_2 = 0$, again we have the very same equations as in GR

$$\Delta V_{1i} - 16\pi G a^2 (\rho + p) \delta v_i = 0; \quad (6.2)$$

$$\partial_{(i} V_{1j)}' + 2\mathcal{H}\partial_{(i} V_{1j)} = 0 \quad (6.3)$$

with $V_{2i} = \mathcal{V}_{12i} = 0$. Clearly, as in GR no vector propagates. In the tensor sector four modes propagates. From the canonical analysis, 8 DOF are expected but only $5 = 1 + 4$ are accounted for. Thus, a scalar plus a vector are not present and are probably strongly coupled, at least around a FRW background. Strong coupling was also found in the Stuckelberg approach [24]. This is not very surprising, condition (3.3) in flat space is equivalent to set to zero the graviton mass and, as a consequence, both spherically symmetric and FRW branch one solutions show no gravity modification [16, 20] and the extra DoF are frozen. Interestingly enough, in the Stuckelberg approach where the second metric is non-dynamical, only the branch one is available and strong coupling is unavoidable, another manifestation of the rather constrained nature of a theory with a priori given metric.

7 Branch Two: Perturbations in dS Phase

Before analyzing branch two case in full generality, it is instructive to consider a particular limit of it: a de Sitter (dS) background, for which we have

$$\begin{aligned} \rho = \text{const.} \Rightarrow \xi = \text{const.} \Rightarrow c = 1, \quad f_1 = f_2, \quad \mathcal{H}_\omega = \mathcal{H} \equiv H a, \\ \text{with } H = \frac{m}{\sqrt{3}\kappa} \sqrt{12\xi(2a_4\xi + a_3) + 2a_2 + \frac{f_1}{\xi^2}}. \end{aligned} \quad (7.1)$$

The dS phase is a fixed point of the FRW geometry of the branch two solution [20]. Introducing

$$\begin{aligned} \Phi_1 &= \frac{1}{2}(\Phi_+ + \Phi_-), & \Phi_2 &= \frac{1}{2\kappa\xi^2}(\Phi_+ - \Phi_-), \\ \Psi_1 &= \frac{1}{2}(\Psi_+ + \Psi_-), & \Psi_2 &= \frac{1}{2\kappa\xi^2}(\Psi_+ - \Psi_-); \end{aligned} \quad (7.2)$$

after some tedious computations one can show that all the equations in the scalar sector are equivalent to a single second order equation for Φ_-

$$\begin{aligned} \Phi_-'' + 2\mathcal{H}\Phi_-' & \left[\frac{2k^4}{9a^2\mathcal{H}^2 m_\Phi^2 + k^4 - 18\mathcal{H}^4} - 1 \right] + \\ & \frac{1}{3}\Phi_- \left[\frac{4(k^6 - 3k^4\mathcal{H}^2)}{9a^2\mathcal{H}^2 m_\Phi^2 + k^4 - 18\mathcal{H}^4} + 3a^2 m_\Phi^2 - k^2 - 6\mathcal{H}^2 \right] = 0; \end{aligned} \quad (7.3)$$

where

$$m_\Phi^2 = m^2 f_1 \left(\frac{1}{\kappa \xi^2} + 1 \right) \quad (7.4)$$

is the mass of the scalar field Φ_- . We have also replaced all space derivatives ∂_m with ik^m and $k^2 = k^i k^j \delta_{ij}$.

All remaining fields can be written in terms of Φ_- :

$$\begin{aligned} \Phi_+ = \Psi_+ = 0, \quad \Psi_- &= \frac{1}{2\mathcal{H}} (2\Phi_-' - m^2 a^2 f_1 \mathcal{B}_1); \\ \mathcal{B}_1 &= \frac{\Phi_- (\kappa \xi^2 + 1) (2k^2 + 3a^2 m_\Phi)}{3a^2 \kappa m_\Phi^2 \xi^2 \mathcal{H}} - \frac{2k^2 \mathcal{E}}{3\mathcal{H}} + 2\mathcal{E}'; \\ \mathcal{E} &= \frac{9\mathcal{H} (\kappa \xi^2 + 1) (a^2 m_\Phi^2 - 2\mathcal{H}^2) \Phi_-' + (\kappa \xi^2 + 1) [3(k^2 + 3\mathcal{H}^2) (a^2 m_\Phi^2 - 2\mathcal{H}^2) + 2k^4] \Phi_-}{2a^2 \kappa \xi^2 m_\Phi^2 (9a^2 \mathcal{H}^2 m_\Phi^2 + k^4 - 18\mathcal{H}^4)}. \end{aligned} \quad (7.5)$$

As a result, just a single scalar DoF propagates.

For what concerns the vector sector, all vectors can be expressed in terms of \mathcal{V}_{12i}

$$V_{2i} = -\frac{m_\Phi^2 a^2 \mathcal{V}_{12i}'}{(1 + \xi^2 \kappa)(a^2 m_\Phi^2 + k^2)}, \quad V_{1i} = -\kappa \xi^2 V_{2i}, \quad (7.6)$$

with \mathcal{V}_{12i} satisfying a second order equation

$$\mathcal{V}_{12i}'' + \frac{2\mathcal{H}\mathcal{V}_{12i}' (2k^2 + a^2 m_\Phi^2)}{k^2 + a^2 m_\Phi^2} + \mathcal{V}_{12i} (k^2 + a^2 m_\Phi^2) = 0. \quad (7.7)$$

In the tensor sector two modes are propagating (4 DoF). The combination $h_{+ij}^{TT} = h_{1ij}^{TT} + \xi^2 \kappa h_{2ij}^{TT}$ is massless and satisfies

$$h_{+ij}^{TT}'' + 2\mathcal{H}h_{+ij}^{TT}' + k^2 h_{+ij}^{TT} = 0. \quad (7.8)$$

The previous equation is the same for tensor perturbations in GR. While for the orthogonal combination $h_{-ij}^{TT} = h_{1ij}^{TT} - \xi^2 \kappa h_{2ij}^{TT}$, we get

$$h_{-ij}^{TT}'' + 2\mathcal{H}h_{-ij}^{TT}' + (k^2 + a^2 m_\Phi^2) h_{-ij}^{TT} + a^2 m_\Phi^2 \frac{(\xi^2 \kappa - 1)}{(\xi^2 \kappa + 1)} h_{+ij}^{TT} = 0. \quad (7.9)$$

Summarizing, in the dS phase we have one scalar, one vector and two tensors that propagate, for a total of $1_S + 2_V + 2_T + 2_T = 7$ DoF, showing that all the expected DoF from the canonical analysis are propagating at perturbative level. We stress that this is not the case for the branch one perturbations. As we will see in the next section, the dynamics of branch two is similar except the presence of a matter fluid provides an additional scalar DoF.

8 Branch Two Perturbations

The dynamics of branch two perturbations is similar to the dS phase, only more involved being ξ not constant and $c \neq 1$. Due to the complexity of the equations it is difficult and physically not very interesting to study them for generic values of ξ . As shown in [20] and summarized in section 3, from early times to redshift of order one, the massive gravity FRW background solutions are characterized by a small value of ξ . As a result, when $\xi \ll 1$, the background solutions can be expanded in series of the dimensionless ratio $\tau/\tau_U \ll 1$, where τ_U is the age of the universe in conformal time. For instance, in the radiation dominated era

$$a = \frac{\tau}{\tau_U} + \epsilon \frac{a_0}{10} \left(\frac{\tau}{\tau_U} \right)^5, \quad (8.1)$$

where $\epsilon = \frac{m^2}{8\pi G \rho_0} = \frac{1}{3} m^2 \tau_U^2$ ⁵. In Appendix B, we give the explicit expressions for the scale factor a including the leading and next to leading terms for both a radiation and matter dominated universe.

An interesting aspect to discuss is the $m \rightarrow 0$ limit. Naively, taking the formal limit $m \rightarrow 0$ in the equations of motion of section (5) we get the perturbed Einstein equations for two separated GR copies. Among the perturbations of metric g that is coupled with matter, one scalar (induced by the presence matter) plus two transverse and traceless tensors propagate. In the sector of the perturbations of \tilde{g} that have no matter sources, only two tensor modes propagate. However, when we take into account that, using the background equations, ξ has a non trivial dependence on m , the very same limit is less straightforward. Indeed, the branch two background solutions are characterized by a value of ξ proportional⁶ to m^2 and this completely changes the nature of the limit $m \rightarrow 0$.

In the equations for the g perturbations, see for instance (5.1-5.3), the coupling with the ones of \tilde{g} is through the effective coupling: $m^2 a^2 f_1$. In the small ξ regime, we have

$$m^2 a^2 f_1 \approx a^2 a_1 m^2 \xi \approx a^2 \frac{a_1^2 m^4}{8\pi G \rho_m} \xrightarrow{m \rightarrow 0} 0. \quad (8.2)$$

Thus, in the small m limit, the perturbations of the metric g precisely coincide with the corresponding in GR. For the equations that govern the perturbation of \tilde{g} , see for instance (5.5-5.7), the effective coupling with g is $\frac{m^2 a^2 f_1}{\kappa \xi^2}$. Then we have that

$$\frac{m^2 a^2 f_{1/2}}{\kappa \xi^2} \approx \frac{a^2 a_1 m^2}{\xi} \approx a^2 8\pi G \rho_m \xrightarrow{m \rightarrow 0} \text{finite}. \quad (8.3)$$

As a result, \tilde{g} perturbations are not GR like in the $m \rightarrow 0$ limit, moreover, as we will show they exhibit a non trivial structure in the momentum k that is very different from the linear structure in k^2 of GR. This peculiar behaviour stems from the interplay of the branch two background and the structure of perturbed Einstein equations; we will return later on this point.

8.1 Scalar perturbations

In the scalar sector, all the fields \mathcal{E} , \mathcal{B}_1 and $\Psi_{1/2}$ are non dynamical and can be expressed in terms of $\Phi_{1/2}$ that satisfy two second order equations; thus 2 scalar DoF propagate. Let us consider the case

⁵If we believe that our theory is origin of Dark Energy we have to take as reference value $m \sim 10^{-33} \text{ eV} \sim H_0$ so that $\epsilon \sim \frac{\Omega_\Lambda}{\Omega_m}$.

⁶Recall that at leading order $\xi = \frac{a_1 m^2}{8\pi G \rho}$.

of a radiation dominated universe. The equations of motion for the two propagating scalars have the following structure

$$\Phi_a'' + \frac{1}{\tau} \mathcal{D}_{ab} \Phi_b' + \frac{1}{\tau^2} \mathcal{M}_{ab} \Phi_b = 0 \quad a, b = 1, 2, \quad (8.4)$$

where \mathcal{D}_{ab} and \mathcal{M}_{ab} are functions of the following dimensionless arguments $\frac{\tau}{\tau_U}$, ϵ and $x = k\tau$. Note that well in the radiation era (and also matter era), we have $\frac{\tau}{\tau_U} \ll 1$; thus, we can expand such a complicated expressions obtaining at leading order

$$\Phi_1'' + \frac{4}{\tau} \Phi_1' + \frac{k^2}{3} \Phi_1 + \mathcal{O}\left(\frac{\tau}{\tau_U}\right) = 0; \quad (8.5)$$

$$\begin{aligned} \Phi_2'' + \frac{10x^2 + 42}{\tau(x^2 + 3)} \Phi_2' + \frac{-5x^6 - 15x^4 + 333x^2 + 999}{3\tau^2(x^2 + 3)^2} \Phi_2 - \\ \frac{36}{\tau(x^2 + 3)} \Phi_1' - \frac{3(5x^2 + 39)}{\tau^2(x^2 + 3)} \Phi_1 + \mathcal{O}\left(\frac{\tau}{\tau_U}\right) = 0. \end{aligned} \quad (8.6)$$

The full expressions are lengthy and not particularly illuminating and are given in Appendix E up to the next to the leading order. We note the various functions \mathcal{D} and \mathcal{M} admit a formal expansion in power of ϵ (i.e m) equivalent on dimensional grounds to an expansion in power of τ/τ_U . Clearly, the equation for Φ_1 is the same than GR plus small corrections, while the equation for Φ_2 has a non trivial $k\tau$ structure whose origin is the effective coupling with the metric g proportional to m^2/ξ that does not vanish in the limit $m \rightarrow 0$. In order to get some physical insight, let us consider the case $x \ll 1$ that physically corresponds to modes well outside the horizon. We have that

$$\Phi_1 \sim \text{const}; \quad (8.7)$$

$$\Phi_2'' + \frac{14}{\tau} \Phi_2' + \frac{37\Phi_2 - 39\Phi_1}{\tau^2} = 0 \Rightarrow \Phi_2 \sim \frac{39}{37} \Phi_1 = \text{const}. \quad (8.8)$$

Thus both scalar perturbations are frozen for the modes well outside the horizon. On the other hand, in the opposite limit, $x \gg 1$, e.g. for the modes well inside the horizon, we have

$$\Phi_1 \sim \frac{1}{k^2 \tau^2} \cos k\tau; \quad (8.9)$$

$$\Phi_2'' + \frac{10}{\tau} \Phi_2' - \frac{5k^2}{3} \Phi_2 - \frac{36}{k^2 \tau^3} \Phi_1' - \frac{15}{\tau^2} \Phi_1 = 0, \Rightarrow \Phi_2 \sim \frac{1}{(k\tau)^{1/2}} e^{+(\frac{5}{3})^{1/2} k\tau} + \mathcal{O}(\Phi_1). \quad (8.10)$$

The solution for the homogeneous part of the Φ_2 equation has runaway exponentially behavior. Such a tachyonic instability is due to the sign of the coefficient of Φ_2 , positive for super-horizon perturbations (and then stable) and negative for sub-horizon perturbation (and then unstable). For the matter dominate case (see Appendix D.1) the situation is very similar and the same kind of instability for the sub-horizon modes is present. Notice that such an instability is not present in dS case, see eq. (7.4).

The leading contribution in the coefficient of Φ_2 for sub-horizon modes ($x \gg 1$) can be computed for a generic equation of state w , in the small ξ limit. The result is

$$\mathcal{M}_{22}|_{x \rightarrow \infty} = \left[-(1 + 2w) + \frac{2(a_0(w + 1)\kappa - 2a_2)}{a_1} \xi + \mathcal{O}(\xi^2) \right] k^2. \quad (8.11)$$

Thus, sub-horizon instabilities are present only when $w > -\frac{1}{2}$ and the Φ_2 perturbation grows exponentially as $e^{(1+2w)k\tau}$. So, for $w > -\frac{1}{2}$, the exponential growth of Φ_2 invalidate perturbation theory

at time $\tau \sim 1/k$. As a consequences, already in the radiation dominated era sub-horizon perturbations become non perturbative, in sharp contrast with GR where matter perturbations become large only when the universe is non relativistic due to Jeans instability. The other scalar fields are given as a function of $\Phi_{1,2}$ in Appendix C.

8.2 Vector perturbations

In the vector sector, using (4.13), as in GR, the velocity perturbation can be easily obtained

$$\delta v_i = \delta v_{0i} a^{3w-1}, \quad (8.12)$$

with δv_{0i} an arbitrary function of k . From (5.8-5.11) one can show that all vectors can be expressed in terms of \mathcal{V}_{12} that satisfies a second order equation given in Appendix F. Thus, only the vector \mathcal{V}_{12} propagates. For instance, in the case of a radiation dominated universe we have at the leading order

$$\delta v = \delta v_0(k) = \text{constant in time}; \quad (8.13)$$

$$V_1 = -\frac{8}{k^2 \tau^2} \delta v_0; \quad (8.14)$$

$$V_2 = \frac{5}{(k^2 \tau^2 + 5)} \mathcal{V}'_{12} - \frac{40}{k^2 \tau^2 (k^2 \tau^2 + 5)} \delta v_0; \quad (8.15)$$

$$\mathcal{V}''_{12} + \frac{8k^2 \tau^2 + 50}{\tau(k^2 \tau^2 + 5)} \mathcal{V}'_{12} + \frac{3}{\tau^2} (k^2 \tau^2 + 5) \mathcal{V}_{12} - \frac{48k^2 \tau^2 + 320}{k^2 \tau^3 (k^2 \tau^2 + 5)} \delta v_0 = 0. \quad (8.16)$$

Then for super horizon modes with $k\tau \ll 1$ we get

$$\mathcal{V}_{12} = C_1 \tau^{-\frac{9}{2} - \frac{\sqrt{21}}{2}} + C_2 \tau^{\frac{1}{2}(\sqrt{21}-9)} + \frac{\delta v_0}{k^2} \left[\frac{64}{7\tau} - \frac{16\tau k^2}{125} \right]; \quad (8.17)$$

where $C_{1,2}$ are arbitrary functions of k . For any reasonable choice of δv_0 there is no growing mode. The structure of the equations is similar in the case of matter dominated universe.

8.3 Tensor perturbations

The evolution equations for tensor perturbations in the radiation era at next to leading order are

$$h_1'' + \frac{2}{\tau} h_1' + k^2 h_1 + \epsilon \frac{\tau^4}{\tau_U^4} \left(\frac{4a_0}{5\tau} h_1' \right) = 0; \quad (8.18)$$

$$h_2'' + \frac{10}{\tau} h_2' + 25k^2 h_1 + \frac{15}{\tau^2} (h_1 - h_2) + \epsilon \frac{\tau^4}{\tau_U^4} \left[-\frac{4(a_0 \kappa - 36a_2)}{5\kappa\tau} h_2' - \frac{162a_2}{\kappa\tau^2} h_1 + \left(\frac{162a_2}{\kappa\tau^2} - \frac{40k^2(a_0\kappa - 6a_2)}{\kappa} \right) h_2 \right] = 0. \quad (8.19)$$

Tensor perturbations h_1 of g behave as in GR, while the one of \tilde{g} , h_2 , beside a sizable coupling with h_1 at early times, show a larger damping factor ($\frac{2}{\tau} \rightarrow \frac{10}{\tau}$) and an effective larger mass ($k^2 \rightarrow 25k^2$). In figure 1 we show the numerical solution of (8.18, 8.19). Clearly there are two regimes, depending on the value of $k\tau$. At very early times, when $k\tau \ll 1$, the two fields $h_{1,2}$ are almost equal and constant due to the large coupling proportional to $(h_1 - h_2)/\tau^2$ present in (8.19). As soon as $k\tau \sim 1$, h_1 starts to oscillate and triggers the oscillations of the h_2 sector (and not the opposite!). Notice that the damping of h_2 is much larger than the one of h_1 , indeed

$$h_1 \sim \frac{1}{\tau}, \quad \text{while} \quad h_2 \sim \frac{1}{\tau^5}. \quad (8.20)$$

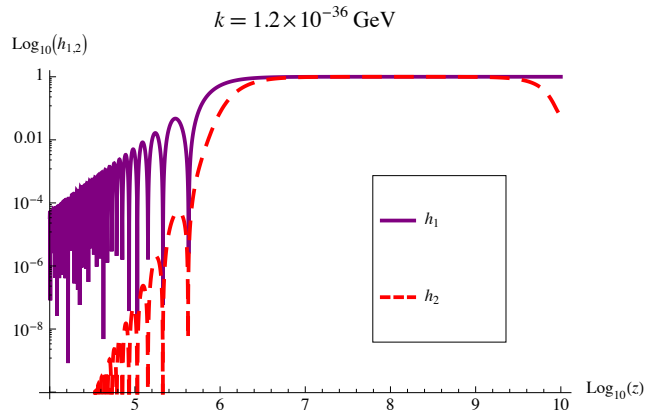


Figure 1. Tensor perturbations $h_{1,2}$ in a radiation dominated universe. The initial conditions are set in deep radiation era ($\Omega_m = 0.2$, $H = 2 \cdot 10^{-42} \text{ GeV}$) at $z = 10^{10}$ as $h_1 = 1$, $h_2 = 1/4$, $h'_{1,2} = 0$.

During the matter era, the results are similar.

Summarizing, h_1 follows closely GR perturbations; on superhorizon scales ($k\tau \ll 1$) h_2 is proportional to h_1 , while sub-horizon modes are greatly suppressed ($h_2 \ll h_1$ when $k\tau \gg 1$). No instabilities are present.

9 Conclusions

In this paper we studied cosmological perturbations around FRW background solutions in a non-linear ghost free massive gravity theory. To construct a massive deformation an auxiliary metric is mandatory. In the Stuckelberg approach the extra metric is taken to be a non-dynamical Minkowski metric. Besides having an “absolute” object that take us back to the time of æther-like theories, such a theories are rather rigid. For instance, there is no black hole solution featuring modified Newton potential [13, 16] and there is no spatially flat FRW solution [17]-[20], thus a spatially negative curvature is required [19, 20]. Even if we allow spatial curvature, the resulting solutions feature an effectively turned off graviton mass [16, 20]. Thus, it is not very surprising that we found the perturbations of the branch one are exactly the same as GR. The extra modes present in the theory have zero kinetic term and do not propagate in a FRW background, this is typical of strong coupling and their propagation is expected at higher order. Perturbation theory around branch one solutions cannot be trusted due to the presence of the strongly coupled extra modes. In the Stuckelberg approach branch one solutions are the only available and nothing more can be said, at least using perturbation theory.

When the second metric is dynamical, the branch two opens up. In this case perturbations are much richer: all 7+1 expected modes propagate, and the reliability of perturbation theory results is a non-trivial issue. In de Sitter all the extra modes propagate and no instability is present at all the length scales. In matter or radiation dominated period, the background solutions is characterized by a small ξ value proportional to the graviton mass m . Such a dependence is the origin of deviation from GR in $m \rightarrow 0$ limit with a non trivial k dependence. In particular, the perturbations of the metric g that couples with matter are very similar to the corresponding ones in GR, while \tilde{g} perturbations are

not GR-like in the $m \rightarrow 0$ limit. Specifically, we found well behaved perturbations for all vector and tensor modes, while one scalar mode shows an exponential Jeans-like instability already well in the radiation epoch, as soon as its wave length enters inside the horizon. Thus, though g perturbations are well behaved, such a fast growth drives sub-horizon scalar perturbations into a non perturbative regime just after a few Hubble times.

A possible way out might be the introduction of a mirror (dark) matter sector minimally coupled to the metric \tilde{g} . As pointed out in the footnote 3, the small ξ regime is the source of instabilities and it can be avoided by the presence of mirror matter reestablishing the normal power counting for small m . The pressure of mirror fluid can contrast the development of sub-horizon instabilities. Of course, the analysis of such a bi-metric theory plus matter and mirror matter is a totally different ballpark and it requires a dedicated investigation that will not be given here, we limit to stress that such an approach would probably not pass the Occam's razor test.

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A Useful Relations

One can verify that the following relations among the additional gauge invariant scalars hold

$$\mathcal{B}_2 = \mathcal{B}_1 - (1 - c^2)\mathcal{E}' ; \quad (\text{A.1})$$

$$\mathcal{H}_\omega \mathcal{F}_2 - \mathcal{H} \mathcal{F}_1 = (\mathcal{H} - \mathcal{H}_\omega)(\Phi_1 - \Phi_2) ; \quad (\text{A.2})$$

$$c^2(\mathcal{F}_2 + \mathcal{F}_1) = (\mathcal{B}_1 - \mathcal{E})(\mathcal{H} + \mathcal{H}_\omega) - 2c^2(\Phi_1 - \Phi_2) ; \quad (\text{A.3})$$

$$c^2(\mathcal{A}_2 - \mathcal{A}_1) = (\mathcal{B}_1 - \mathcal{E}') [c(\mathcal{H} - \mathcal{H}_\omega) - c'] ; \quad (\text{A.4})$$

$$c^2(\mathcal{A}_2 + \mathcal{A}_1) = 2c^3(\Psi_2 - \Psi_1) + \mathcal{B}_1 [c(\mathcal{H}_\omega + \mathcal{H}) - 3c'] + 3c'\mathcal{E}' \quad (\text{A.5})$$

$$+ c [2\mathcal{B}'_1 - 2\mathcal{E}'' - \mathcal{E}'(\mathcal{H}_\omega + \mathcal{H})] . \quad (\text{A.6})$$

B FRW Background

- During matter era, in the small ξ regime, we have $\rho = \rho_0/a^3$, $\rho_0 \equiv \frac{12}{8\pi G \tau_U^2}$ and for $\tau/\tau_U \ll 1$

$$a = \frac{\tau^2}{\tau_U^2} + \frac{a_0 \epsilon \tau^8}{7 \tau_U^8} + \frac{\tau^{14}}{\tau_U^{14}} \frac{3 \epsilon^2 (4 a_0^2 \kappa + 49 a_1^2)}{637 \kappa} + \dots \quad (\text{B.1})$$

- During radiation era, in the small ξ regime, we have $\rho = \frac{\rho_0}{a^4}$, $\rho_0 \equiv \frac{3}{8\pi G \tau_U^2}$ and for $\tau/\tau_U \ll 1$

$$a = \frac{\tau}{\tau_U} + \epsilon \frac{a_0 \tau^5}{10 \tau_U^5} + \epsilon^2 \frac{(a_0^2 \kappa + 20 a_1^2)}{120 \kappa} \frac{\tau^9}{\tau_U^9} + \dots \quad (\text{B.2})$$

C Scalar Perturbations in the Radiation Era

For completeness we give also the leading expressions for the remaining scalars that can be expressed in terms of $\Phi_{1/2}$

$$\Psi_1 = -\Phi_1, \quad \Psi_2 = -\frac{3\mathcal{E}}{5\tau^2} - \Phi_2; \quad (\text{C.1})$$

$$u_s = \frac{1}{2}\tau(\tau\Phi'_1 + \Phi_1), \quad \frac{\delta\rho}{\rho} = \frac{2}{3}\Phi_1(k^2\tau^2 + 3) + 2\tau\Phi'_1; \quad (\text{C.2})$$

$$\mathcal{B}_2 = -60\frac{\mathcal{E}}{\tau} - 20\tau(\tau\Phi'_2 + 5\Phi_2); \quad (\text{C.3})$$

$$\mathcal{E} = -\frac{\tau^2(-9\Phi_1(k^2\tau^2 + 9) + \Phi_2(2k^4\tau^4 + 15k^2\tau^2 + 99) + 9\tau(\Phi'_2 - 3\Phi'_1))}{3(k^2\tau^2 + 3)^2}. \quad (\text{C.4})$$

D Scalar Perturbations in the Matter Era

Leading order of the scalar evolution equations during matter era

$$\Phi''_1 + \frac{6}{\tau}\Phi'_1 = 0; \quad (\text{D.1})$$

$$\begin{aligned} \Phi''_2 + \frac{4(4x^4 + 81x^2 + 720)}{\tau(x^4 + 18x^2 + 144)}\Phi'_2 + \frac{-x^6 + 18x^4 + 960x^2 + 11808}{\tau^2(x^4 + 18x^2 + 144)}\Phi_2 - \\ \frac{6(x^4 + 36x^2 + 432)}{\tau(x^4 + 18x^2 + 144)}\Phi'_1 - \frac{x^6 + 78x^4 + 1728x^2 + 14688}{\tau^2(x^4 + 18x^2 + 144)}\Phi_1 = 0. \end{aligned} \quad (\text{D.2})$$

E Next to Leading Corrections for Scalars in Radiation

For completeness here we give the next to leading correction for the scalar equations of motion (8.4).

$$\mathcal{D}_{11} = 4 + \epsilon \frac{\tau^4}{\tau_U^4} \frac{8a_0}{45} + \epsilon^2 \frac{\tau^8}{\tau_U^8} \frac{4 \left(2a_0^2 \kappa (x^2 + 3)^2 + 25a_1^2 (4x^4 + 132x^2 + 117) \right)}{6075 \kappa (x^2 + 3)^2}; \quad (\text{E.1})$$

$$\mathcal{D}_{12} = \epsilon^2 \frac{\tau^8}{\tau_U^8} \frac{2a_1^2 (5x^4 + 6x^2 + 27)}{9 (x^2 + 3)^2 \kappa}; \quad (\text{E.2})$$

$$\mathcal{D}_{21} = \frac{36}{x^2 + 3} + \epsilon \frac{\tau^4}{\tau_U^4} \frac{4 \left(3a_0 \kappa (x^4 + 9x^2 - 72) + 10a_2 (2x^6 + 24x^4 + 90x^2 + 81) \right)}{15 \kappa (x^2 + 3)^3}; \quad (\text{E.3})$$

$$\begin{aligned} \mathcal{D}_{22} &= \frac{10x^2 + 42}{x^2 + 3} \\ &- \epsilon \frac{\tau^4}{\tau_U^4} \frac{4a_0 (5x^6 + 42x^4 + 108x^2 + 351) + 40a_2 (4x^6 + 36x^4 + 108x^2 + 81)}{45 \kappa (x^2 + 3)^3}; \end{aligned} \quad (\text{E.4})$$

$$\mathcal{M}_{11} = \frac{x^2}{3} + \epsilon \frac{\tau^4}{\tau_U^4} \frac{4a_0}{9x^2} + \epsilon^2 \frac{\tau^8}{\tau_U^8} \frac{4a_0^2 \kappa (x^2 + 3)^2 + 45a_1^2 (3x^4 + 38x^2 + 15)}{405 \kappa x^2 (x^2 + 3)^2}; \quad (\text{E.5})$$

$$\mathcal{M}_{12} = \epsilon^2 \frac{\tau^8}{\tau_U^8} \frac{a_1^2 (2x^6 + 83x^4 + 96x^2 + 459)}{9 (x^2 + 3)^2 \kappa}; \quad (\text{E.6})$$

$$\mathcal{M}_{21} = 15 + \frac{72}{x^2 + 3} \quad (\text{E.7})$$

$$- \epsilon \frac{\tau^4}{\tau_U^4} \frac{36a_0 \kappa (x^2 - 27) (x^2 + 9) + 10a_2 (8x^8 + 177x^6 + 1305x^4 + 3807x^2 + 3159)}{45 \kappa (x^2 + 3)^3} \quad (\text{E.8})$$

$$\mathcal{M}_{22} = \frac{333 - 5x^4}{3x^2 + 9} + \epsilon \frac{\tau^4}{\tau_U^4} \left(\frac{4a_0 (10x^8 + 75x^6 - 54x^4 - 1323x^2 - 8748)}{405 (x^2 + 3)^3} + \right. \quad (\text{E.9})$$

$$\left. \frac{+5a_2 (-16x^8 + 51x^6 + 1899x^4 + 8181x^2 + 7533)}{405 \kappa (x^2 + 3)^3} \right). \quad (\text{E.10})$$

F Vector Perturbations

Equation of motion of the vector propagating mode

$$\frac{2f_2 \kappa k^2 \xi^3 \mathcal{H}}{J} \mathcal{V}_{12}'' + \frac{2\kappa k^2 \xi^2 \mathcal{N}_1}{J^2} \mathcal{V}_{12}' + f_1 k^2 \mathcal{V}_{12} + \mathcal{N}_0 \delta v = 0; \quad (\text{F.1})$$

where

$$\begin{aligned} \mathcal{N}_1 &= 2a^2 f_2 m^2 [2\mathcal{H}^3 (\kappa \xi^4 + \xi^2) + 4\xi \mathcal{H}^2 \xi' + \xi \xi' \mathcal{H}' + \mathcal{H} (3\xi'^2 - \xi \xi'')] \\ &+ \kappa k^2 \xi^2 [f_2 (4\xi \mathcal{H}^2 \xi' + 8\xi^2 \mathcal{H}^3 + \xi \xi' \mathcal{H}' + \mathcal{H} (\xi'^2 - \xi \xi'')) + \xi \mathcal{H} f_2' \mathcal{H}_\xi] , \end{aligned} \quad (\text{F.2})$$

and

$$\begin{aligned} J^2 \mathcal{N}_0 &= 16\pi G m^2 a^2 \kappa \xi^2 (w + 1) \rho_m \{ 2a^2 f_2^2 m^2 [\xi' (\xi (2\mathcal{H}^2 + \mathcal{H}') + 3\mathcal{H} \xi') - \xi \mathcal{H} \xi''] \\ &+ \kappa f_2 k^2 \xi^2 [4\xi^2 \mathcal{H}^3 + 2\xi \mathcal{H}^2 \xi' + \xi \xi' \mathcal{H}' + \mathcal{H} (\xi'^2 - \xi \xi'')] + \kappa k^2 \xi^3 \mathcal{H} \mathcal{H}_\xi f_2' \} . \end{aligned} \quad (\text{F.3})$$

All vectors are expressed in terms of \mathcal{V}_{12} , indeed, omitting the spatial indices, we get

$$\chi = \frac{\kappa k^2 \xi^2 \mathcal{V}'_{12} \mathcal{H}_\xi}{K} - \frac{16\pi a^2 \delta v G \kappa \xi^2 (w+1) \mathcal{H} \xi \rho}{K}, \quad \mathcal{H}_\xi = \xi' + 2\xi \mathcal{H}; \quad (\text{F.4})$$

$$K = 2m^2 a^2 f_2 [\xi' + \mathcal{H} (\kappa \xi^3 + \xi)] + \kappa k^2 \xi^2 \mathcal{H}_\xi;$$

and

$$V_1 = -\frac{16\pi G a^2 (w+1) \rho \delta v [2m^2 a^2 f_2 (c\xi \mathcal{H}) + \kappa k^2 \xi^2 \mathcal{H}_\xi]}{K k^2} - \frac{2\kappa m^2 a^2 f_2 \xi^3 \mathcal{H} \mathcal{V}'_{12}}{K}, \quad (\text{F.5})$$

$$V_2 = \frac{2m^2 a^2 f_2 (\xi' + \xi \mathcal{H}) \mathcal{V}'_{12}}{K} - \frac{32m^2 \pi G a^4 f_2 (w+1) \rho (c\xi \mathcal{H}) \delta v}{K k^2}.$$

References

- [1] V. A. Rubakov and P. G. Tinyakov, *Phys. Usp.* **51** (2008) 759. arXiv:0802.4379 [hep-th].
K. Hinterbichler, arXiv:1105.3735 [hep-th].
- [2] C. de Rham, G. Gabadadze, *Phys. Rev.* **D82**, 044020 (2010). arXiv:1007.0443 [hep-th].
- [3] M. Fierz and W. Pauli, *Proc. Roy. Soc. Lond. A* **173**, 211 (1939).
- [4] H. van Dam and M. J. G. Veltman, *Nucl. Phys.* B **22** (1970) 397.
Y. Iwasaki, *Phys. Rev.* D **2** (1970) 2255;
V.I.Zakharov, *JETP Lett.* **12** (1971) 198.
- [5] A. I. Vainshtein, *Phys. Lett. B* **39**, 393 (1972).
- [6] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, *Annals Phys.* **305**, 96 (2003). arXiv:hep-th/0210184.
- [7] C. de Rham, G. Gabadadze, A. J. Tolley, *Phys. Rev. Lett.* **106**, 231101 (2011). arXiv:1011.1232 [hep-th].
- [8] D. G. Boulware and S. Deser, *Phys. Lett. B* **40**, 227 (1972).
- [9] S. F. Hassan and R. A. Rosen, *Phys. Rev. Lett.* **108** (2012) 041101. arXiv:1106.3344 [hep-th].
S. F. Hassan, R. A. Rosen and A. Schmidt-May, arXiv:1109.3230 [hep-th].
- [10] C. de Rham, G. Gabadadze, A. J. Tolley, arXiv:1107.3820 [hep-th].
- [11] T. Damour and I. I. Kogan, *Phys. Rev. D* **66** (2002) 104024.
- [12] C. J. Isham, A. Salam and J. A. Strathdee, *Phys. Rev. D* **3**, 867 (1971);
A. Salam and J. A. Strathdee, *Phys. Rev. D* **16** (1977) 2668.
- [13] C. Deffayet, T. Jacobson, [arXiv:1107.4978 [gr-qc]].
- [14] L. Berezhiani, G. Chkareuli, C. de Rham, G. Gabadadze and A. J. Tolley, arXiv:1111.3613 [hep-th].
- [15] Z. Berezhiani, D. Comelli, F. Nesti and L. Pilo, *JHEP* **0807** (2008) 130. arXiv:0803.1687 [hep-th].
- [16] D. Comelli, M. Crisostomi, F. Nesti, L. Pilo, *Phys. Rev. D* **85** (2012) 024044. arXiv:1110.4967 [hep-th].
- [17] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, A.J. Tolley, *Phys. Rev. D* **84** (2011) 124046. arXiv:1108.5231 [hep-th].
- [18] L. Alberte, arXiv:1110.3818 [hep-th].
- [19] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, *JCAP* **1111**, 030 (2011). arXiv:1109.3845 [hep-th].
- [20] D. Comelli, M. Crisostomi, F. Nesti and L. Pilo, arXiv:1111.1983 [hep-th].

- [21] M. von Strauss, A. Schmidt-May, J. Enander, E. Mortsell and S. F. Hassan, arXiv:1111.1655 [gr-qc].
- [22] M. S. Volkov, JHEP **1201** (2012) 035, arXiv:1110.6153 [hep-th].
- [23] K. Koyama, G. Niz and G. Tasinato, JHEP **1112** (2011) 065. arXiv:1110.2618 [hep-th].
- [24] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, arXiv:1111.107 [hep-th].
- [25] Z. Berezhiani, D. Comelli, F. Nesti and L. Pilo, Phys. Rev. Lett. **99**, 131101 (2007).
arXiv:hep-th/0703264.
- [26] D. Blas, D. Comelli, F. Nesti and L. Pilo, Phys. Rev. D **80**, 044025 (2009). arXiv:0905.1699 [hep-th].
- [27] S. F. Hassan and R. A. Rosen, arXiv:1109.3515 [hep-th].